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An Existence Theorem for a General Goursat Problem

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1. INTRODUCTION

Let $u(x, y)$ be a real-valued function defined in some rectangular neighbourhood V of the origin in R^2. The derivative of u with respect to x is denoted by u_x and the derivative of u with respect to y by u_y. The mixed derivative with respect to x and y is denoted by u_{xy}. These derivatives are supposed to exist and to be continuous together with u. The function $f(x, y, z_1, z_2, z_3)$ is bounded and continuous in $V \times R^3$.

We look at the problem of finding *a* u as above such that

$$u_{xy} = f(x, y, u, u_x, u_y), \qquad u(0, y) = u(x, 0) = 0, \qquad (x, y) \in V. \quad (1.1)$$

If f is Lipschitz continuous in the z-variables then it is a classical result that the solution of (1.1) exists and is unique. See the proof by E. Picard in Darboux [*2*] or the proof in Kamke [*6*].

If f is independent of z_2 and z_3 then it follows that (1.1) has at least one solution u, Montel [*9*].

Leehey [*8*] and Hartman and Wintner [*4*], have proved that if f is Lipschitz continuous in z_2 and z_3 then it follows that (1.1) has at least one solution. Sharper results have been obtained by Alexiewiecz and Orlicz [*1*], who weakened the hypothesis of continuity of f to some extent. Walter [*12*] kept the hypothesis of continuity but substituted the Lipschitz continuity in z_2 and z_3 by a weaker condition. Recently Santagati [*11*], has proved existence theorems that covers all such theorems mentioned above. He also treats data given in a more general way. A generalization of the classical result with Lipschitz continuity in all z-variables can be found in Persson [*10*]. See Theorem 2 in [*10*] or Theorem 1 in Section 3 of this paper. Note that there is no hypothesis that f is bounded. Also note that the proof applies equally well to a bounded rectangular region as to all of R^n. Theorem 3 in [*10*] is a generalization of the Montel theorem in [*9*]. See also Theorem 2 in Section 3 below.

In [*8*] and [*4*] it has been pointed out that there is no uniqueness in general in the theorem by Montel mentioned above. In [*4*] it is also proved that mere continuity of f in (1.1) is not enough to guarantee the existence of a solution of (1.1).

The present aim is to generalize the result in Leehey [*8*], and in Hartman and Winter [*4*] mentioned above. By doing so we restrict ourselves to continuous functions with Lipschitz conditions on some variables. The generalization is contained in Theorem 3 in section 3.

A part of the first rather long proof of Theorem 3 used a generalization of the majorization used in Walter [*12*]. For that part Professor L. Gårding suggested a more dirct procedure. However it included iteration of nonzero data. The proof was still rather cumbersome to write down and to justify in the general case. The situation suggested that some exponential majorization analogous to that used in [*10*] might remove the local nature of the Gårding procedure. Thus it turned out that the proof could be made considerably shorter by a proper exponential majorization. This variant will be presented in Section 3. The rest of the proof uses Theorem 2 in [*10*] and the technique used in the proof of Theorem 3 in [*10*].

For further details concerning (1.1) the reader is referred to Diaz [*3*], Walter [*12*], and Santagati [*11*], and to the references given in these papers. A special case of Theorem 1 below is treated by Kovač [*7*]. He treats the case $n = 2$, $\beta = (t, t)$.

The notation and some definitions used in Section 3 are introduced in Section 2. Section 3 contains Theorems 1, 2, and 3 together with the proof of Theorem 3.

2. Preliminaries

Let $x = (x_1, ..., x_n) \in R^n$ and $z = (z_1, ..., z_N) \in C^N$. By $\alpha = (\alpha_1, ..., \alpha_n)$ we denote a multi-index with non-negative integers as components. If $D_x = D = (\partial/\partial x_1, ..., \partial/\partial x_n)$, then we write $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. We also write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|x| = |x_1| + \cdots + |x_n|$, $\alpha! = \alpha_1! \cdots \alpha_n!$, and

$$\alpha \leqslant \beta \Leftrightarrow \alpha_j \leqslant \beta_j, \qquad 1 \leqslant j \leqslant n.$$

Definition. *Let $u(x)$ be a complex-valued function defined in all R^n, and let β be a multi-index. If all derivatives $D^\alpha u$, $\alpha \leqslant \beta$, exist and are continuous together with u itself, then we say that u belongs to the function class $C(\beta, R^n)$.*

Definition. Let $u \in C(\beta, R^n)$. We define $u = O(x^\beta)$ by

$$u = O(x^\beta) \Leftrightarrow D_j^k u(x) = 0, \qquad x_j = 0, \qquad 0 \leqslant k < \beta_j, \qquad 1 \leqslant j \leqslant n.$$

DEFINITION. *The function $f(x, z)$ is a complex-valued continuous function defined in all $R^n \times C^N$. If to every compact subset $K \subset R^n$ there exists a constant M such that*

$$|f(x, z) - f(x, z')| \leqslant M\,|z - z'|, \qquad x \in K, \qquad z \in C^N, \qquad z' \in C^N,$$

then we say that f belongs to the function class $CL(0, R^n \times C^N)$. Here we have let

$$|z - z'| = |z_1 - z_1'| + \cdots + |z_N - z_N'|.$$

3. THEOREMS FOR GENERAL GOURSAT PROBLEMS

We start by restating Theorem 2 in [*10*].

THEOREM 1. *Let f belong to $CL(0, R^n \times C^N)$, and let β and α^k, $1 \leqslant k \leqslant N$, be multi-indices such that*

$$\alpha^k \leqslant \beta, \qquad \alpha^k \neq \beta, \qquad 1 \leqslant k \leqslant N.$$

Then it follows that the problem

$$D^\beta u = f(x, D^{\alpha^1}u, \ldots, D^{\alpha^N}u), \qquad u = O(x^\beta).$$

has a unique solution $u \in C(\beta, R^n)$.

Theorem 3 in [*10*] is reformulated in the following way.

THEOREM 2. *Let $f(x, z)$ be a continuous complex-valued function defined in $R^n \times C^N$. The function f is supposed to have compact support in $R^n \times C^N$. The multi-indices α^k, $1 \leqslant k \leqslant N$, are restricted by*

$$\alpha_j^k < \beta_j, \qquad 1 \leqslant j \leqslant n, \qquad 1 \leqslant k \leqslant N.$$

It follows that the problem

$$D^\beta u = f(x, D^{\alpha^1}u, \ldots, D^{\alpha^N}u), \qquad u = O(x^\beta),$$

has at least one solution $u \in C(\beta, R^n)$.

The proof of theorem 2 is a part of the proof of Theorem 3 in [*10*], so it will not be repeated here. A comparison between Theorems 1 and 2 above and the corresponding literature for the special case $n = 2$, $\beta = (1,1)$, suggests that the following theorem is true.

THEOREM 3. *Let $f(x, z)$ be a complex-valued function defined in $R^n \times C^N$*

having compact support. There exist an integer N', $1 \leqslant N' \leqslant N$, *and a constant* $M > 0$ *such that*

$$|f(x, z_1, ..., z_{N'}, z_{N'+1}, ..., z_N) - f(x, z_1, ..., z_{N'}, \bar{z}_{N'+1}, ..., \bar{z}_N)|$$
$$\leqslant M \sum_{k=N'+1}^{N} | z_k - \bar{z}_k |, \qquad (x, z_1, ..., z_{N'}, z_{N'+1}, ..., z_N) \in R^n \times C^N,$$
$$(x, z_1, ..., z_{N'}, \bar{z}_{N'+1}, ..., \bar{z}_N) \in R^N \times C^N. \tag{3.1}$$

Let β *and* α^k, $1 \leqslant k \leqslant N$, *be multi-indices such that*

$$\alpha_j^k < \beta_j, \qquad 1 \leqslant j \leqslant n, \qquad 1 \leqslant k \leqslant N', \tag{3.2}$$

and also such that

$$\alpha^k \leqslant \beta, \qquad \beta \neq \alpha^k, \qquad 1 \leqslant k \leqslant N. \tag{3.3}$$

It follows that the Goursat problem

$$D^\beta u = f(x, D^{\alpha^1}u, ..., D^{\beta^N}u), \qquad u = O(x^\beta), \tag{3.4}$$

has at least one solution $u \in C(\beta, R^n)$.

Proof of Theorem 3. As in the proof of Theorem 2 f is regularized as a function defined in R^{n+2N}. See Hörmander [5], p. 3, and [10]. We get infinitely differentiable functions f^ϵ such that

$$f^\epsilon(x, z) = \int f(x - y, z - z')\, g_\epsilon(y, z')\, dy\, dz'. \tag{3.5}$$

The functions g_ϵ are infinitely differentiable nonnegative functions with compact support. They are also such that

$$\int g_\epsilon(y, z')\, dy\, dz' = 1, \epsilon > 0.$$

The functions f^ϵ tends to f uniformly in all $R^n \times C^N$ when ϵ tends to zero. It follows from (3.5) and from (3.1) that

$$|f^\epsilon(x, z_1, ..., z_{N'}, z_{N'+1}, ..., z_N) - f^\epsilon(x, z_1, ..., z_{N'}, \bar{z}_{N'+1}, ..., \bar{z}_N)|$$
$$\leqslant \int |f(x - y, z_1 - z_1', ..., z_N - z_N')$$
$$- f(x - y, z_1 - z_1', ..., \bar{z}_N - z_N')|\, g_\epsilon(y, z')\, dy\, dz'$$
$$\leqslant \left(M \sum_{k=N'+1}^{N} | z_k - \bar{z}_k |\right) \int g_\epsilon(y, z')\, dy\, dz' = M \sum_{N'<k\leqslant N} | z_k - z_k' |. \tag{3.6}$$

The functions f^ϵ are infinitely differentiable with compact support. Therefore they must form a subset of $CL(0, R^n \times C^N)$. Theorem 1 now says that there is a unique function $u^\epsilon \in C(\beta, R^n)$ such that

$$D^\beta u^\epsilon = f^\epsilon(x, D^{\alpha^1}u^\epsilon, \ldots, D^{\alpha^N}u^\epsilon), \qquad u^\epsilon = O(x^\beta). \tag{3.7}$$

We now define

$$f_m = f^{m^{-1}}, \qquad \text{and} \qquad u_m = u^{m^{-1}}.$$

The Arzelà theorem will be applied to the sequence $(D^\beta u_m)_1^\infty$. We shall pick out a subsequence that converges to a function v. It will follow that the function u determined by $D^\beta u = v$, $u = O(x^\beta)$ is a solution of (3.4). The main difficulty is to prove that $(D^\beta u_m)_1^\infty$ is equicontinuous. So we are now going to prove this.

There is a constant K such that

$$| D^\beta u_m(\bar{x})| \leqslant \sup | f_m(x, z)| \leqslant \sup | f(x, z)| \leqslant K, \qquad \bar{x} \in R^n.$$

Let negative powers of D denote integration. It follows from above and from $u_m = O(x^\beta)$, that to every $R > 0$ there exists a new constant K such that

$$| D^{-\eta} D^\beta u_m(x)| \leqslant K, \quad \eta \leqslant \beta, \quad | x | \leqslant R, \quad m = 1, 2, 3, \ldots . \tag{3.8}$$

We now choose a fixed R such that $| x | \geqslant R$ implies that $D^\beta u_m(x) = 0$, $m = 1, 2, 3, \ldots$. This is possible since f has compact support. See (3.5), and [5], p. 3. As a first step we shall prove that the set

$$\{D^{\alpha^k} u_m , m = 1, 2, 3, \ldots, \alpha_1{}^k < \beta_1 , 1 \leqslant k \leqslant N\}$$

is equicontinuous in the x_1-variable. Let $\eta_1 = \beta_1 - \alpha_1{}^k - 1$, and $\eta_j = \beta_j - \alpha_j{}^k$, $2 \leqslant j \leqslant n$. It follows from (3.3) that $\eta \geqslant 0$. Let $(x_1 , x'') \in R^n$ and $(x_1' , x'') \in R^n$. From (3.8) we may now conclude that

$$\begin{aligned} &| D^{\alpha^k} u_m(x_1 , x'') - D^{\alpha^k} u_m(x_1' , x'')| \\ &\leqslant \left| \int_0^{x_1} D^{-\eta} D^\beta u_m(t_1 , x'')\, dt_1 - \int_0^{x_1'} D^{-\eta} D^\beta u_m(t_1 , x'')\, dt_1 \right| \\ &= \left| \int_{x_1'}^{x_1} D^{-\eta} D^\beta u_m(t_1 , x'')\, dt_1 \right| \\ &\leqslant K| x_1 - x_1' |, \qquad |(x_1 , x'')| \leqslant R, \qquad |(x_1' , x'')| \leqslant R. \end{aligned} \tag{3.9}$$

By (3.9) it is now proved that the set under consideration is equicontinuous and also that the functions of the set are Lipschitz continuous with the same Lipschitz constant. We shall need that information in the following.

If $\alpha_1^k = \beta_1$, then (3.2) implies that $N' < k \leqslant N$. For such a k we let $\eta^k = \beta - \alpha^k$. We write somewhat symbolically

$$\int_{0,\eta^k}^{x''} D^\beta u_m(x_1, x''(t''))\,dt'' = D^{-\eta^k} D^\beta u_m(x_1, x'') = D^{\alpha^k} u_m(x_1, x'').$$

Here $x''(t'')_j = x_j$ if $\eta_j^k = 0$. If $\eta_j^k \neq 0$ there is one or more integrations in the jth variable. This is symbolically denoted by $x''(t'')_j = t_j$.

Let $x''(t'') = y''$ for a moment. Then we obtain

$$\begin{aligned} B &= D^\beta u_m(x_1, y'') - D^\beta u_m(x_1', y'') \\ &= f_m((x_1, y''), D^{\alpha^1} u_m(x_1, y''),\dots, D^{\alpha^N} u_m(x_1, y'')) \\ &\quad - f_m((x_1', y''), D^{\alpha^1} u_m(x_1', y''),\dots, D^{\alpha^N} u_m(x_1', y'')). \end{aligned}$$

This difference is rewritten as a sum of differences. Dots denote those variables which have the same value in the two terms of each difference.

$$\begin{aligned} B &= f_m((x_1, y''),\dots) - f_m((x_1', y''),\dots) \\ &\quad + f_m(\cdot, D^{\alpha^1} u_m(x_1, y''),\dots) - f_m(\cdot, D^{\alpha^1} u_m(x_1', y''),\dots) \\ &\quad + \cdots + f_m(\dots, D^{\alpha^N} u_m(x_1, y'')) - f_m(\dots, D^{\alpha^N} u_m(x_1', y'')). \end{aligned}$$

It follows from the uniform continuity of f and from (3.5) that there exists a function $d'(\delta) \geqslant 0$, $\delta \geqslant 0$ such that $d'(\delta)$ tends to zero when δ tends to zero, and such that

$$| f_m(x, z) - f_m(y, z')| \leqslant d'(\delta), \quad |(x - y, z - z')| \leqslant \delta, \quad m = 1, 2, 3,\dots .$$

This combined with (3.9) and (3.6) gives

$$\begin{aligned} | B | &\leqslant d(| x_1 - x_1' |) + N d'(K | x_1 - x_1' |) \\ &\qquad + M \sum_{\eta_1^k = 0} | D^{\alpha^k} u_m(x_1, y'') - D^{\alpha^k} u_m(x_1', y'')|, \\ &\qquad |(x_1, y'')| \leqslant R, \qquad |(x_1', y'')| \leqslant R. \end{aligned} \tag{3.10}$$

Let $d(\delta) = d'(\delta) + Nd'(K\delta)$. We rewrite (3.10) as

$$| B | \leqslant d(| x_1 - x_1' |) + M \sum_{\eta_1^k=0} \left| \int_{0,\eta^k}^{x''} (D^\beta u_m(x_1, x''(t'')) - D^\beta u_m(x_1', x''(t''))) \, dt'' \right|. \tag{3.11}$$

Let $| x'' | = | x_2 | + \cdots + | x_n |$. We define

$$d_m(\delta) = \sup_{\substack{|x_1 - x_1'| \leqslant \delta \\ |x''| \leqslant R}} | D^\beta u_m(x_1, x'') - D^\beta u_m(x_1', x'')| \, e^{-2MN|x''|}, \qquad m = 1, 2, \ldots . \tag{3.12}$$

Combination of (3.12) with (3.11) when $| x_1 - x_1' | \leqslant \delta$ gives

$$| B | \leqslant d(\delta) + M \sum_{\eta_1^k=0} \left| \int_{0,\eta^k}^{x''} d_m(\delta) \, e^{2MN|x''(t'')|} \, dt'' \right|.$$

Note that for $L > 0$

$$\left| \int_0^s e^{L|t|} \, dt \right| = L^{-1}(e^{L|s|} - e^0) \leqslant L^{-1} \, e^{L|s|}.$$

This is the fundamental property of exponential majorization. Combined with the fact that $\eta^k \neq 0$ this now gives

$$| B | \leqslant d(\delta) + Md_m(\delta) \sum_{\eta_1^k=0} (2MN)^{-|\eta^k|} \, e^{2MN|x''|},$$

and

$$| B | \leqslant d(\delta) + 2^{-1} d_m(\delta) e^{2MN|x''|}.$$

In other words, it is now proved that

$$| B | e^{-2MN|x''|} \leqslant d(\delta) e^{-2MN|x''|} + 2^{-1} d_m(\delta) \leqslant d(\delta) + 2^{-1} d_m(\delta).$$

So it follows from (3.12) that

$$d_m(\delta) \leqslant d(\delta) + 2^{-1} d_m(\delta) \Leftrightarrow d_m(\delta) \leqslant 2d(\delta), \qquad m = 1, 2, \ldots .$$

Let

$$d_m'(\delta) = \sup_{\substack{|x_1 - x_1'| \leqslant \delta \\ |x''| \leqslant R}} | D^\beta u_m(x_1, x'') - D^\beta u_m(x_1', x'')|.$$

Thus it is proved that

$$d'_m(\delta) \leqslant e^{2MNR}\, d_m(\delta) \leqslant 2e^{2MNR}\, d(\delta), \qquad m = 1, 2, \ldots .$$

It follows that $(D^\beta u_m)_1^\infty$ is uniformly equicontinuous in x_1. But we may as well prove that this is true for every x_j, $1 \leqslant j \leqslant n$. From that we conclude that $(D^\beta u_m)_1^\infty$ is equicontinuous. This together with (3.9) shows that we may apply the Arzelà theorem. Let v be the continuous limit of a uniformly convergent subsequence of $(D^\beta u_m)_1^\infty$. It will be proved that the function u determined by

$$D^\beta u = v, \qquad u = O(x^\beta),$$

is a solution of (3.4).

We may assume that the subsequence above is identical with the sequence $(D^\beta u_m)_1^\infty$ itself. It then follows that the sequence $(D^\alpha u_m)_1^\infty$ converges uniformly to $D^\alpha u$, when m tends to infinity, if $|x| \leqslant R$ and if $\alpha \leqslant \beta$.

It is now easy to prove that u is a solution of (3.4). In order to prove this we write

$$\begin{aligned} D^\beta u &- f(x, D^{\alpha^1}u, \ldots, D^{\alpha^n}u) \\ &= D^\beta u - D^\beta u_m + f_m(x, D^{\alpha^1}u_m, \ldots, D^{\alpha^N}u_m) - f(x, D^{\alpha^1}u_m, \ldots, D^{\alpha^N}u_m) \\ &\quad + f(x, D^{\alpha^1}u_m, \ldots, D^{\alpha^N}u_m) - f(x, D^{\alpha^1}u, \ldots, D^{\alpha^N}u). \end{aligned}$$

$D^\beta u_m$ tends to $D^\beta u$ uniformly in R^n. The function f is uniformly continuous in $R^n \times C^N$. The sequence $(D^{\alpha^k}u_m)_1^\infty$ converges uniformly to $D^{\alpha^k}u$, $|x| \leqslant R$, $1 \leqslant k \leqslant N$. The sequence $(f_m)_1^\infty$ tends uniformly to f in $R^n \times C^N$. This shows that the right member of the identity tends to zero when m tends to infinity. Then u must be a solution for $|x| \leqslant R$. But u is a solution in all R^n since R can be chosen arbitrarily great. It is obvious that $u \in C(\beta, R^n)$. The proof of Theorem 3 is finished.

Note added in proof. For another generalization of Theorem 2 see Persson, J., Non-characteristic Cauchy problems and generalized Goursat problems in R^n. To appear in *J. Math. Mech.*

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